



# $W_X^{1,p}$ INTERIOR ESTIMATES FOR VARIATIONAL HYPOELLIPTIC OPERATOR WITH $VMO_X$ COEFFICIENTS

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**ABSTRACT.** We consider a divergence form hypoelliptic operator consisting of a system of real smooth vector fields  $X_1, \dots, X_q$  satisfying Hörmander condition in some domain  $\Omega \subseteq \mathbb{R}^n$ . Interior  $L^p$  estimates,  $2 \leq p < \infty$ , can be obtained for weak solutions of the equation  $X_j^T(a^{ij}X_i u) = X_j^T F^j$ , by assuming that the coefficients  $a^{ij}$  belong locally to the space  $VMO_X$  with respect to the Carnot–Caratheodory metric induced by the vector fields.

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## 1. INTRODUCTION AND MAIN RESULT

In the present paper we obtain interior  $L^p$  estimates for weak solutions of the equation  $X_j^T(a^{ij}X_i u) = X_j^T F^j$ , where  $X_1, \dots, X_q$  is a family of real vector fields satisfying Hörmander’s condition in some domain  $\Omega \subseteq \mathbb{R}^n$ , and the coefficients  $a^{ij}$  belong locally to the space  $VMO_X$ , with respect to the Carnot–Caratheodory metric induced by the vector fields. Our result generalizes, to the setting of hypoelliptic variational operators of Hörmander type, the  $L^p$  regularity results previously obtained in [CFL1, DF]. Indeed, in [CFL1, DF] local estimates of this kind for weak solutions of elliptic equation, both in divergence and non divergence form, are obtained by assuming that the coefficients of the operators belong to the space  $VMO$  with respect to the Euclidean setting. More precisely our theorem is the following:

**Theorem 1.** *Let  $X_1, \dots, X_q$  Hörmander vector fields of step  $r$  at each point of a given domain  $\Omega \subseteq \mathbb{R}^n$ ,  $q \leq n$  (we can assume  $n \geq 3$ ); moreover let  $2 \leq p < \infty$ . Let*

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us consider the following variational equality of divergence form:

$$(1.1) \quad X_j^T(a^{ij}X_i u) = \operatorname{div}_X F$$

where, as usual,  $\operatorname{div}_X F \equiv X_j^T F^j$  and moreover,  $u \in W_{loc,X}^{1,p}(\Omega)$  is a weak solution of (1.1) if

$$(1.2) \quad \int_{\Omega} a^{ij}(x)X_i u(x)X_j \phi(x)dx = \int_{\Omega} F^j(x)X_j \phi(x)dx \quad \text{for any } \phi \text{ test in } \Omega.$$

Let us suppose that

- i)  $\{a^{ij}\}_{i,j=1,\dots,q}$  is a symmetric measurable matrix defined in  $\Omega$  such that  $a^{ij} \in VMO_X(B) \cap L^\infty(B)$  for any open Euclidean ball  $B \Subset \Omega$ ;
- ii) there exists  $\nu > 0$  such that  $\frac{1}{\nu}|w|^2 \leq a^{ij}(x)w_i w_j \leq \nu|w|^2$  for any  $w \in \mathbb{R}^q$  and a.e.  $x \in \Omega$ ;
- iii)  $F \in L_{loc}^p(\Omega, \mathbb{R}^q)$ .

Then, for any  $\Omega' \Subset \Omega$  there exists a constant  $c = c(\Omega', \Omega, X_1, \dots, X_q, p, a^{ij}, \nu)$  and there exists an open set  $\Omega'', \Omega' \Subset \Omega'' \Subset \Omega$ , such that

$$(1.3) \quad \|u\|_{W_X^{1,p}(\Omega')} \leq c \left( \|u\|_{L^p(\Omega'')} + \|F\|_{L^p(\Omega'', \mathbb{R}^q)} \right).$$

Note that letter  $c$  denotes a generic constant that can be different also in the same line.

$VMO$  functions, studied by Sarason in [Sa], do appear first in [CFL1, CFL2], in order to obtain  $L^p$  estimates for the solutions of uniformly elliptic equations in non divergence form, and later in [DF], in non divergence form. In both two cases techniques rely on suitable representation formulas, on singular integrals depending on a parameter, and on their commutators with  $BMO$  functions.  $VMO$  condition is a type of discontinuity which implies some kind of average continuity: in such a sense,  $VMO$  hypothesis extends classical theory of Agmon–Douglis–Nirenberg in [ADN1, ADN2]. Indeed uniformly continuous bounded functions, as well the ones in  $W^{1,n}$  and  $W^{1,\frac{n}{\theta}}$ ,  $\theta \in ]0, 1[$ , belong to  $VMO$ .

The introduction of such a family of vector fields goes back to the paper of Hörmander [Ho] where the author shows that hypoellipticity of the solution of a differential equation related to a sum of squares of vector fields follows from a geometric condition on the vector fields and their commutators. Later, Rothschild e Stein in [RS], deal with the problem of a natural setting in which such a sum of square operators can be cast. The algebraic structures that do appear in this new setting are nowadays known as Carnót groups; in particular, Euclidean spaces are a very particular cases. These are particular simply connected nilpotents Lie groups whose finite dimensional Lie algebra admits a graduated stratification in vector subspaces. It follows that this algebraic structure is naturally equipped with a family of automorphisms which generalize the standard product with scalars in  $\mathbb{R}^n$ . Finally, a well known theorem of Rothschild e Stein shows how it is possible to approximate a class of differential operators consisting of a system of Hörmander vector fields, through invariant differential operators defined in suitable Carnót groups. From the metric viewpoint, we can naturally settle these spaces in a general class of metric spaces nowadays known as *Carnót–Caratheodory Metric Spaces*, where metric is introduced through suitable finite families of Lipschitz vector fields. Such metric spaces has been intensively studied in the last thirty years in several setting of pure and applied mathematics such as degenerate elliptic differential equations, hypoelliptic differential operators, sub–Riemannian manifolds, control theory, mathematical models of human vision, robotics, geometric measure theory. In particular, when the vector fields inducing the metric satisfy Hörmander

condition, the associate metric  $d_X$  enjoys many good properties: for instance, the induced topology is actually the Euclidean one; all necessary properties for our purpose can be found in [CarFan]. It should be clear that, in this paper, such a metric will play a role because the coefficients  $a^{ij}$  of the operator belong to the space  $VMO_X$  defined through the Carnot–Caratheodory metric induced by the vector fields associated to the hypoelliptic variational operator.

Coming back to hypoelliptic operators consisting of a family of Hörmander vector fields,  $VMO_X$  functions do appear in the papers of Bramanti and Brandolini [Brm-Brn1, Brm-Brn2]. Indeed, the coefficients  $a^{ij}$  are assumed to belong to the  $VMO_X$  space with respect to the metric induced from the vector fields: clearly in general the space  $VMO_X$  is different from the “Euclidean”  $VMO$ , so particular metric proofs must be adapted in this new setting. Moreover, the proofs of the results in the Euclidean setting (see [CFL1, DF]) need several notions: the existence of a translation invariant fundamental solutions smooth away from the origin, convolution operators, representation formulas, parametrized singular integrals and Riesz potential, commutators with  $BMO$  functions, analysis on spaces of homogeneous type, properties of  $VMO$  functions. In the new setting, and in particular in our case, these notions and proofs can be adapted by employing the technics introduced by Rothschild e Stein, see [Brm-Brn2]) so that, all in all, the properties of the solution of the given equation can be recovered from the properties of the solution of a new equation associated to a new operator defined locally on a suitable Carnot group, pulling back local estimates in this last setting to a local estimate for the solution of the given operator. In particular, arguing as in [Brm-Brn2], the use of a parametrix implies that coefficients  $a^{ij}$  be smooth, that is every function in  $VMO_X$  should be approximated by a sequence of smooth functions: this is actually possible: in [CarFan], in the setting of a general space of homogeneous type, the space  $VMO$  is defined both through balls and “cubes”, and the density property with smooth functions with respect to the  $BMO$  norm is proved, in the particular case of a Carnot–Caratheodory spaces whose metric is associated to a finite family of Hörmander vector fields.

The regularity result and significative properties of the space  $VMO_X$  are contained in the doctoral thesis discussed on December 2002. The regularity result has been announced at the *XVII Congresso U.M.I.* hold in Milan (Italy) on September 8–13, 2003, at the conference “*Aspetti Teorici ed Applicativi di Equazioni alle derivate parziali*” hold in Maiori (Italy) on April 21–24, 2004, and at a talk given in Bologna (Italy) in summer 2004. The density results employed in this paper has been published later in 2007, see [CarFan], and so mentioned in the following.

## 2. PRELIMINARIES

**2.1. Carnot groups and Carnot–Caratheodory metric spaces.** We refer to Section 3 of [CarFan] for basic definition on Carnot groups and Carnot–Caratheodory metric spaces, associated in particular to a family of Hörmander vector fields; in the same section can be found the statement of the Ball–Box theorem, useful for our purpose.

**2.2. Hörmander vector fields: theorem of Rothschild e Stein.** Let  $X_1, \dots, X_q$  smooth real vector fields defined on a smooth manifold. For  $s \in \mathbb{N}$ ,  $i_1, i_2, \dots, i_{s-1}, i_s \in \{1, 2, \dots, q\}$  let  $I = (i_1, i_2, \dots, i_{s-1}, i_s)$  and

$$(2.1) \quad X_I = \left[ X_{i_1}, [X_{i_2}, \dots [X_{i_{s-1}}, X_{i_s}] \dots] \right].$$

We say that  $I$  has *length*  $s$  and we call *commutator of length*  $s$  any vector field such that  $X \in \text{Span}\{X_I\}_{\text{length } I=s}$ ; commutators of length 1 are just the elements of the span of the vector fields  $X_1, \dots, X_q$ . Suppose that for every  $x \in M$  there exists  $s(x) \in \mathbb{N}$  such that  $\text{Span}\{X_I(x)\}_{\text{length } I \leq s(x)} = T_x(M)$ ; then we say that the vector fields  $X_1, \dots, X_q$  *satisfy Hörmander condition* of step  $r \in \mathbb{N}$  if  $s(x) \leq r$  for any  $x \in M$ . We finally recall that the vector fields  $X_1, \dots, X_q$  are *free up to the step*  $r$  *at the point*  $x \in M$  if the vectors  $X_I(x)_{\text{length } I \leq r}$  are linearly independent, except for Jacobi's and anticommutativity relations.

Let now  $\mathfrak{g}_{q,r} = V_1 \oplus \dots \oplus V_r$  be the nilpotent free Lie algebra of step  $r$  with  $q = \dim_{\mathbb{R}}(V_1)$  generators, and let  $\mathbb{G}_{q,r}$  be the corresponding free Carnót group (recall that, from a set-theoretical viewpoint, it is possible to assume that  $\mathbb{G}_{q,r}$  is some  $\mathbb{R}^N$  endowed with a suitable product of Lie group and with a suitable family of dilations  $\{\gamma_s\}_{s>0}$ ). We can denote by  $\{Y_{jk^j}\}_{\substack{1 \leq j \leq r \\ 1 \leq k^j \leq n_j}}$  a base of  $V_j$ , where

$j \in \{1, \dots, r\}$  and  $k^j \in \{1, \dots, n_j\}$  ( $n^j$  is a positive integer depending on  $V_j$ ); for the sake of simplicity we shall denote by  $Y_1, \dots, Y_q$  the generators of the first layer of  $\mathfrak{g}_{q,r}$ ; we can denote by  $(y_{jk^j})$  the exponential coordinates of the first kind of  $y = \exp_1(Y)$  in  $\mathbb{G}_{q,r}$ , where  $Y = \sum_{j=1}^r \sum_{k^j=1}^{n_j} y_{jk^j} Y_{jk^j} \in \mathfrak{g}_{q,r}$  denotes an element of the algebra. Finally let  $\{\delta_s\}_{s>0}$  be the family of automorphisms of  $\mathfrak{g}_{q,r}$ .

Then, the vector fields  $X_1, \dots, X_q$  are free up to the step  $r$  at the point  $x \in M$  if and only if  $\dim_{\mathbb{R}}(\text{Span}\{X_I(x)\}_{\text{length } I \leq r}) = \dim(\mathbb{G}_{q,r})$ , where the last number denotes the dimension of  $\mathbb{G}_{q,r}$  as a smooth manifold.

Let us suppose now that the vector fields do satisfy Hörmander condition of step  $r$  at  $x_0 \in M$ ; let  $n = \dim(M)$ ,  $N = \dim(\mathbb{G}_{q,r})$ ,  $k = N - n$ ,  $\widetilde{M} = M \times \mathbb{R}^k$  and let  $\pi : \widetilde{M} \rightarrow M$  the canonical projection.

Then we have the following “lifting theorem” of Rothschild–Stein.

**Theorem 2.** *Let  $X_1, \dots, X_q$  smooth vector fields defined on  $M$  satisfying Hörmander condition of step  $r$  at the point  $x_0 \in M$ . Then there exist  $\{\lambda_{jl}(x, t)\}_{\substack{1 \leq j \leq q \\ n+1 \leq l \leq N}}$  smooth functions of the new variables  $t_{n+1}, \dots, t_N$ , defined in a neighborhood of  $\xi_0 = (x_0, 0) \in \widetilde{U} = U \times U' \subset \widetilde{M}$ , where  $U$  is a neighborhood of  $x_0$  in  $M$  and  $U'$  a neighborhood of 0 in  $\mathbb{R}^k$ , such that the vector fields*

$$\widetilde{X}_j = X_j + \sum_{l=n+1}^N \lambda_{jl}(x, t) \partial_{t_l}, \quad j = 1, \dots, q,$$

*are free up to the step  $r$  at each point of  $\widetilde{U}$ .  $\square$*

**Remark 1.** It is easy to verify that also the vector fields  $\widetilde{X}_j$  satisfy Hörmander condition of step  $r$  at each point  $\xi \in \widetilde{U}$  and that it results

$$\widetilde{X}_j(f \circ \pi) = X_j f \circ \pi$$

for any  $f \in C^\infty(U)$  and for any  $j = 1, \dots, q$ .

**Notation 1.** For any  $f$  defined in some subset  $S \subseteq \Omega$ , we shall denote by either  $f \circ \pi$  or  $\widetilde{f}$  the function defined in  $S \times \mathbb{R}^k$  that maps  $\xi = (x, t) \in S \times \mathbb{R}^k$  to  $f(x)$ .

Let  $\lambda \in \mathbb{R}$ ,  $\lambda > 0$ . A measurable function  $f : \mathbb{G}_{q,r} \rightarrow \mathbb{R}$  is said to be *homogeneous of degree*  $\lambda$  if  $f \circ \gamma_s = s^\lambda f$  for any  $s > 0$ ; a differential operator  $D$  on  $\mathbb{G}_{q,r}$  is said to be *homogeneous of degree*  $\lambda$  if  $D(f \circ \gamma_s) = s^\lambda (Df) \circ \gamma_s$ , for any  $s > 0$  and for any  $f \in C^\infty(\mathbb{G}_{q,r})$ . Then it immediately follows that if  $D$  and  $f$  are a differential operator and a function, respectively homogeneous of degrees  $\lambda_1$  e  $\lambda_2$ , then  $Df$  e

$fD$  are a function and an operator homogeneous respectively of degrees  $\lambda_2 - \lambda_1$  and  $\lambda_1 - \lambda_2$ .

Let us recall now the notion of local degree at the origin (see [RS] pag.272 and [Brm-Brn2] pag.789). Let  $D$  be a differential operator. We say that  $D$  is *homogeneous of local degree*  $\leq \lambda$  if all Taylor polynomials of the coefficients of the operator give rise, up to a rearrangements, to a sum of differential operators of degree at most  $\lambda$  in  $\mathbb{G}_{q,r}$ .

Let us suppose now  $\widetilde{X}_1, \dots, \widetilde{X}_q$  be free vector fields up to the step  $r$  at a point  $\xi_0$  of a smooth manifold  $\widetilde{M}$ . Then  $\dim_{\mathbb{R}}(\text{Span}\{\widetilde{X}_I(\xi_0)\}_{\text{length } I \leq r}) = \dim(\mathbb{G}_{q,r})$ . Now, if all  $\widetilde{X}_j$  were invariant on  $\mathbb{G}_{q,r}$ , it would be possible to identify them with the elements of (the first layer of)  $\mathfrak{g}_{q,r}$  and, consequently, recover the elements of  $\mathbb{G}_{q,r}$  through the before mentioned exponential coordinates; this is not possible in general because the vector fields are not invariant in general; nevertheless, if we choose  $\{\widetilde{X}_{jk^j}\}$  such that  $\text{Span}\{\widetilde{X}_I(\xi_0)\}_{\text{length } I \leq r} = T_{\xi_0}(\widetilde{M})$ , we can consider the mapping that, for any  $N$ -tuple of real numbers  $y = (y_{jk^j})$  in a fixed closed ball  $B$  around 0 sufficiently small, maps any  $\xi$  in a compact neighborhood  $\widetilde{U}'$  of  $\xi_0$  to the element  $\eta = \exp_{\widetilde{X}}(1)$ ,  $\widetilde{X} = \sum_{j=1}^r \sum_{k^j=1}^{n_j} y_{jk^j} \widetilde{X}_{jk^j}$  : indeed, under these conditions, we have the existence, unicity and  $C^\infty$  dependence from  $\xi$  ed  $y$  of the Cauchy problem  $\phi'(t) = \widetilde{X}(\phi(t))$  and  $\phi(0) = \xi$ , for  $t$  in a sufficiently small neighborhood of 0, and for any fixed  $\xi \in \widetilde{U}'$ ,  $y \in B$ . Then we have  $\phi(t) = \exp\left(t\left(\sum_{j=1}^r \sum_{k^j=1}^{n_j} y_{jk^j} \widetilde{X}_{jk^j}\right)\right)\xi$ ; in particular  $\eta = \phi(1) = \exp\left(\sum_{j=1}^r \sum_{k^j=1}^{n_j} y_{jk^j} \widetilde{X}_{jk^j}\right)\xi$  and

$$y = \Theta_\xi(\eta) \quad \text{e} \quad \eta = \Theta_\xi^{-1}(y), \quad \text{for any } \xi, \eta \in \widetilde{U}', y \in B.$$

The mapping  $\Theta_\xi$  then behaves like a coordinate map; indeed through  $\Theta_\xi$  we can think of  $\widetilde{X}_j$  as defined on  $\mathbb{G}_{q,r}$  and consequently to approximate them with the left invariant vector fields  $Y_j$  on  $\mathbb{G}_{q,r}$  (which, in their turn, can be chosen so that they agree with the  $j$ -th partial derivatives at the origin (see [RS] pag. 272; see also [SC] for the following formulation); we can also think of the  $y = (y_{jk^j})$  as a system of canonical coordinates, depending only on the vector fields  $\{\widetilde{X}_{jk^j}\}_{\substack{1 \leq j \leq r \\ 1 \leq k^j \leq n_j}}$ .

Then we have the following approximation theorem of Rothschild and Stein.

**Theorem 3.** *Let  $\widetilde{X}_1, \dots, \widetilde{X}_q$  real smooth vector fields defined on a smooth manifold  $\widetilde{M}$  and let  $\xi_0 \in \widetilde{M}$ . Let us suppose that the vector fields  $\widetilde{X}_1, \dots, \widetilde{X}_q$  do satisfy the Hörmander hypothesis of step  $r$ ; moreover we assume that the vector fields are free of step  $r$  at the same point  $\xi_0$ . Let us choose the vector fields  $\{\widetilde{X}_{jk^j}\}_{\substack{1 \leq j \leq r \\ 1 \leq k^j \leq n_j}}$  as before and let us denote by  $(y_{jk^j})$  the associated system of canonical coordinates. Let  $\mathfrak{g}_{q,r}$  and  $\mathbb{G}_{q,r}$  respectively the free Lie algebra with  $q$  generators of step  $r$  and the associated free Carnot group.*

*Then it is possible to choose a base  $\{Y_{jk^j}\}_{\substack{1 \leq j \leq r \\ 1 \leq k^j \leq n_j}}$  of  $\mathfrak{g}_{q,r}$  such that  $Y_j(0) = \frac{\partial}{\partial y_j}$ ,  $j = 1, \dots, q$ , a neighborhood  $U$  of 0 in  $\mathbb{G}_{q,r}$ , two open neighborhood  $W, V$  of  $\xi_0$  in  $\widetilde{M}$ ,  $W \subseteq V$ , such that the following facts hold:*

- i)  $\Theta_\xi|V$  is a diffeomorphism between  $V$  and  $\Theta_\xi(V)$  for any  $\xi \in V$ ;
- ii)  $\Theta_\xi(V) \supseteq U$  for any  $\xi \in W$ ;
- iii) the mapping  $\Theta : V \times V \rightarrow \mathbb{G}_{q,r}$  defined by the position  $\Theta(\xi, \eta) = \Theta_\xi(\eta)$  belongs to  $C^\infty(V \times V)$ ;
- iv) for any fixed  $\xi \in W$  the mapping  $\eta \rightarrow \Theta_\xi(\eta) = \Theta(\xi, \eta) = (y_{jk^j})$ ,  $\eta \in W$ , is a coordinate map for  $W$  and  $(\Theta_\xi)_* \widetilde{X}_i = Y_i + R_i^\xi$  in  $U$ , dove  $R_i^\xi$  is a real

smooth vector fields of local degree  $\leq 0$ , with  $C^\infty$  dependence on  $\xi \in W$ ; more precisely it means that for any  $\xi \in W$  and for any  $f \in C^\infty(\mathbb{G}_{q,r})$  it results

$$(2.2) \quad \widetilde{X}_i \circ (f \circ \Theta_\xi) = (Y_i \circ f) \circ \Theta_\xi + (R_i^\xi \circ f) \circ \Theta_\xi.$$

In general, for any couple of indexes  $j$  e  $k^j$  and,

$$(\Theta_\xi)_* \widetilde{X}_{jk^j} = Y_{jk^j} + R_{jk^j}^\xi$$

where  $R_{jk^j}^\xi$  is a real smooth vector field of local degree  $\leq j-1$  with  $C^\infty$  dependence on  $\xi \in W$ .  $\square$

Recall that  $(\Theta_\xi)_*$  is the mapping induced by  $\Theta_\xi$  on the fiber bundle and defined by the position  $((\Theta_\xi)_* \widetilde{X})f = (\widetilde{X} \circ (f \circ \Theta_\xi)) \circ \Theta_\xi^{-1}$ , for any vector field and for any  $f \in C^\infty(V)$ .

**2.3. Introduction of a quasi-metric equivalent to C-C metric.** In what follows we set  $M = \Omega$  and  $\widetilde{M} = \Omega \times \mathbb{R}^k \subseteq \mathbb{R}^N$ , then our neighborhoods are  $C$ - $C$  balls which are open sets in any one of the topologies  $\tau_{Euclidean}$  and  $\tau_{C-C}$ . We shall denote by  $Q$  the homogeneous dimension as a doubling spaces in  $\mathbb{G}_{q,r}$ . It will be more useful to introduce a quasi-metric  $\rho$  in  $V$ , equivalent to  $d_{\widetilde{X}}$ . Indeed, referring the reader to [SC] and [Brm-Brn2], let us recall main definitions.

**Theorem 4.** *Let  $V$  and  $W$  neighborhood of  $\xi_0$  as in Theorem (3). If we set*

$$\rho(\xi, \eta) = \|\Theta(\xi, \eta)\|_{\mathbb{G}_{q,r}} \quad \text{for any } \xi, \eta \in V$$

*then the following properties hold*

- i)  $\Theta(\xi, \eta) = \Theta(\eta, \xi)^{-1} = -\Theta(\eta, \xi)$ ;
- ii)  $\rho(\xi, \eta) \leq c(\rho(\xi, \zeta) + \rho(\eta, \zeta))$  for any  $\xi, \eta \in V$  such that  $\rho(\xi, \eta) \leq 1$  and  $\rho(\xi, \zeta) \leq 1$ ;
- iii) *there exist four positive smooth functions  $V \ni \xi \rightarrow \zeta(\xi), \omega(\xi)$ ,  $V \ni \eta \rightarrow h(\eta)$ ,  $U \ni y \rightarrow j(y)$  and a positive constant  $c$  such that  $1/c \leq \zeta, \omega, h, j \leq c$  respectively on  $V$  the first three ones, on  $U$  the last one, and  $J_\xi(\eta) = \zeta(\xi)h(\eta)$  e  $J_\xi^{-1}(y) = \omega(\xi)j(y)$ , where  $J_\xi(\eta)$  and  $J_\xi^{-1}(y)$  are respectively the jacobians of the mappings  $y = \Theta_\xi(\eta)$  and  $\eta = \Theta_\xi^{-1}(y)$ .  $\square$*

Let now  $x_0 \in \Omega$ , and  $\xi_0 = (x_0, 0) \in \widetilde{\Omega}$ ; we can assume that  $V = B \times R$  where  $B$  is an open Euclidean ball around  $x_0$  and  $I$  is an open rectangle in  $\mathbb{R}^k$ ; consequently we can consider the following three quasi-metric spaces with respective Lebesgue measures

$$(2.3) \quad (B, d_X, dx), \quad (V, d_{\widetilde{X}}, d\xi), \quad (V, \rho, d\xi).$$

Then from Theorem (4) we have the following proposition.

**Proposition 1.** *According to above notation, for any  $\xi_0 = (x_0, 0) \in \widetilde{\Omega}$ , there exists an Euclidean neighborhood  $V$  of the type  $B \times I$ , where  $B$  is an Euclidean open ball around  $x_0$  and  $I$  is an open rectangle in  $\mathbb{R}^k$  around 0, such that  $(V, \rho, d\xi)$  is a bounded doubling space. In particular, Lebesgue measure  $d\xi \equiv m_N$  is a doubling measure and, for any open ball  $B_\rho \subset V$  related to the metric  $\rho$  we have  $m_N(B_\rho) \approx r^Q$ .  $\square$*

Finally, as in Lemma 7 of [SC] we have the following proposition (see [NaStWa2] for the proof).

**Proposition 2.** *According to above notation, there exists  $c > 0$  such that*

$$\frac{1}{c}\rho(\eta, \xi) \leq d_{\widetilde{X}}(\eta, \xi) \leq c\rho(\eta, \xi) \quad \text{per ogni } \eta, \xi \in V. \quad \square$$



So from the metric viewpoint, the metric spaces  $(V, d_{\tilde{X}}, d\xi)$  and  $(V, \rho, d\xi)$  are equivalent. Let us now compare the spaces  $BMO_X(B)$  (resp.  $VMO_X(B)$ ) defined in  $B$  with respect to the metric induced from the vector fields  $X_1, \dots, X_q$ , endowed with the corresponding Lebesgue measure, with the spaces  $BMO_{\tilde{X}}(V)$  (resp.  $VMO_{\tilde{X}}(V)$ ) of functions defined on  $B$  with respect to the metric induced by the vector fields  $\tilde{X}_1, \dots, \tilde{X}_q$ , and endowed with the corresponding Lebesgue measure. The arguments are taken from pagg. 793–794 of [Brm-Brn2], then we just recall the following propositions.

**Proposition 3.** *According to above notation, for any  $x \in \Omega$  and  $r > 0$  such that  $B_{d_X}(x, r) \subset B$  and  $B_{d_{\tilde{X}}}(\xi, r) \subset V$ , if  $\pi : \tilde{\Omega} \rightarrow \Omega$  and  $\pi_k : \tilde{\Omega} \rightarrow \mathbb{R}^k$  are the canonical projections, denoting by  $m_n, m_k, m_N$  respective Lebesgue measures, and setting  $C_k = m_k(\pi_k(B_{d_{\tilde{X}}}(\xi, r)))$ , it results*

- $\pi(B_{d_{\tilde{X}}}(\xi, r)) = B_{d_X}(x, r)$ ;
- $d_{\tilde{X}}(\xi, \xi') \geq d_X(x, x') \quad \forall x, x' \in \Omega$ ;
- *there exists  $C > 0$  such that, for  $r$  small enough,  $\frac{1}{CC_k} m_N(B_{d_{\tilde{X}}}(\xi, r)) \leq m_n(B_{d_X}(x, r)) \leq \frac{C}{C_k} m_N(B_{d_{\tilde{X}}}(\xi, r))$ .  $\square$*

**Proposition 4.** *According to above notation, if  $f : \Omega \rightarrow \mathbb{R}$  is a measurable function, then  $f \in BMO_X(B)$  (resp.  $f \in VMO_X(B)$ ) with respect to metric  $d_X$  and Lebesgue measure  $m_n$  if and only if  $f \circ \pi \in BMO_{\tilde{X}}(V)$  (resp.  $f \circ \pi \in VMO_{\tilde{X}}(V)$ ) with respect to metric  $\rho$  Lebesgue measure  $m_N$ .  $\square$*

**2.4. Differential operators, fundamental solutions and parametrices.** In this section our space of homogeneous type will be  $(V, d_{\tilde{X}}, d\xi) \equiv (V, \rho, d\xi)$ , according to notations of Section (2.3). We recall results of Sections 2.1. e 3.2. in [Brm-Brn2]. Let  $L$  a given differential operator consisting of a family  $X_0, X_1, \dots, X_q$  of Hörmander vector fields defined on a given open set  $\Omega \subset \mathbb{R}^n$ . For instance let either  $L = \sum_{i=1}^n X_i^2 + X_0$ , or  $L = a^{ij} X_i X_j$ , where the coefficients belong to  $C^\infty(\Omega)$ . Arguing as in [RS] we will recover properties of the operator  $L$  from the properties of a new operator  $\tilde{L}$ , consisting of the  $\tilde{X}_0, \tilde{X}_1, \dots, \tilde{X}_q$ ; this last operator, in its turn, has much more properties useful for our purpose because it can be written as a sum of two more operators: the first one consists of left invariant vector fields defined on a suitable nilpotent group  $\mathbb{G}$ , the second one, defined in  $\mathbb{G}$ , is a smooth operator of local degree equal to zero so that results of Folland (see Teorema 2.1, Section 2 in [Fo1]) can be applied, for the existence of a fundamental solution. Finally, quoting Christ (see [Chr2], Example 8, Pag.96) we need some more analysis to obtain our estimates; more precisely we need to construct, through a suitable coordinate map (see Theorem (3)), two parametrices (see §15. in [RS]) that behave, in our case, as left and right partial inverse of the operator  $\tilde{L}$ , up to a finite number of operators (depending only on  $\tilde{L}$ ) which, in their turn, are the analogous of classical integral with either singular kernel, or fractional or Riesz potential. So, given the estimates with new vector fields  $\tilde{X}_0, \tilde{X}_1, \dots, \tilde{X}_q$ , we can recover the original estimate (see Remark (1)). In our case, the operator is  $L = X_j^T(a^{ij} X_i)$  where, as before,  $X_1, \dots, X_q$  is a family of Hörmander vector fields defined on a given open set  $\Omega \subset \mathbb{R}^n$ , whose coefficients belong locally to the class  $VMO_X$  with respect to  $C$ – $C$  metric induced from the vector fields  $X_1, \dots, X_q$ ; we consider weak solutions for the equation. Then, through vector fields  $\tilde{X}_1, \dots, \tilde{X}_q$ , we pass from the equation associated to the operator  $X_j^T(a^{ij} X_i)$  to the one associated to the divergence form operator  $\tilde{X}_j^T(\tilde{a}^{ij} \tilde{X}_i)$  consisting of the vector fields  $\tilde{X}_1, \dots, \tilde{X}_q$  (note that, in this case, thanks to Proposition (4), coefficients  $\tilde{a}^{ij}$  belong locally to  $VMO_{\tilde{X}}$  with respect to  $C$ – $C$  metric induced by vector fields  $\tilde{X}_1, \dots, \tilde{X}_q$ ; if we are able to obtain

local estimate for the solution of the new equation, we can obtain the requested estimates. So we need estimates for the solutions of the equation associated to the operator  $\tilde{L} = \tilde{X}_j^T (\tilde{a}^{ij} \tilde{X}_i)$  with coefficients locally in  $VMO_{\tilde{X}}$  with respect to  $C$ - $C$  metric associated to vector fields  $\tilde{X}_1, \dots, \tilde{X}_q$ . Now, thanks to results of Section (2.6), it is possible to approximate locally coefficients  $\tilde{a}^{ij}$  with smooth functions; then the divergence form operator agrees, up to some low order terms which belong to the span of the vector fields  $\tilde{X}_1, \dots, \tilde{X}_q$ , with the non divergence form hypoelliptic one (see Teorema 1.11 in [Brm-Brn1]). For the main part of this operator we can consider the parametrix adapted by the authors in [Brm-Brn2] from the original one of Rothschild–Stein in [RS], pag. 296. Then, by assuming that coefficients  $\tilde{a}^{ij}$  are smooth, we can obtain a first estimate for a test solution  $u$ : this is the argument of *Step 2* of pag. 11.

Let us now recall the results in the form useful for this purpose.

**Theorem 5.** *Let  $X_1, \dots, X_q$  be Hörmander vector fields, let  $\tilde{X}_1, \dots, \tilde{X}_q$  be the free vector fields associated and let  $Y_1, \dots, Y_q \in \mathfrak{g}_{q,r}$  the approximating left invariant vector fields, according to Theorem (3). Then, for any fixed  $\xi_0 \in \mathbb{G}_{q,r}$ , the operator  $\tilde{a}^{ji}(\xi_0)Y_jY_i$  is hypoelliptic jointly with its transposed. Under this conditions there exists  $\Gamma_0 \equiv \Gamma_{\xi_0} \in C^\infty(\mathbb{G}_{q,r} \setminus \{0\})$ , homogeneous in  $\mathbb{G}_{q,r}$  of degree  $2 - Q$  and such that for any test function  $\phi$  in  $\mathbb{G}_{q,r}$  and any  $\xi \in \mathbb{G}_{q,r}$  it results*

$$\phi(\xi) = \int_{\mathbb{G}_{q,r}} \Gamma_0(\eta^{-1}\xi) (\tilde{a}^{ji}(\xi_0)Y_jY_i\phi)(\eta) d\eta.$$

Moreover, for any  $i, j = 1, \dots, q$  there exists constants  $\alpha_{ij}(\xi_0)$  such that for any  $\xi \in \mathbb{G}_{q,r}$  we have

$$Y_jY_i\phi(\xi) = \text{P.V.} \int_{\mathbb{G}_{q,r}} Y_jY_i\Gamma_0(\eta^{-1}\xi) (\tilde{a}^{ji}(\xi_0)Y_jY_i\phi)(\eta) d\eta + \alpha_{ji}(\xi_0) (\tilde{a}^{ji}(\xi_0)Y_jY_i\phi)(\xi),$$

$Y_jY_i\Gamma_0 \in C^\infty(\mathbb{G}_{q,r} \setminus \{0\})$ , where  $Y_jY_i\Gamma_0$  is homogeneous in  $\mathbb{G}_{q,r}$  of degree  $-Q$ . Moreover  $\sup_{\xi \in \mathbb{R}^N} |\alpha_{ji}| < \infty$ .  $\square$

According to notations in previous sections let us recall definitions and main properties of “operators of type 0, 1, 2”.

**Definition 1** (Kernel and operators of type 0, 1, 2). *Let  $\xi_0 \in V$  be fixed. We say that  $K_{0,0}(\xi, \eta) \equiv K_{\xi_0,0}(\xi, \eta)$  (resp.  $K_{0,1}(\xi, \eta) \equiv K_{\xi_0,1}(\xi, \eta)$ ,  $K_{0,2}(\xi, \eta) \equiv K_{\xi_0,2}(\xi, \eta)$ ) is a frozen kernel at  $\xi_0$  of Type 0 (resp. 1, 2) if, according to notations of Theorem (5), for any  $m \in \mathbb{N}$ , it can be written as a finite sum of the kind*

$$\begin{aligned} & \left[ a_0(\xi)(D_0\Gamma_0)(\Theta(\xi, \eta))b_0(\eta) \right] + \\ & + \left[ a_1(\xi)(D_1\Gamma_0)(\Theta(\eta, \xi))b_1(\eta) \right] + \dots + \left[ a_s(\xi)(D_s\Gamma_0)(\Theta(\eta, \xi))b_s(\eta) \right] \end{aligned}$$

with  $s = s(m) \in \mathbb{N}$ ,  $a_i, b_i$  test functions in  $V$  for any  $i = 0, 1, \dots, s$ ,  $D_1, \dots, D_s$  differential operators homogeneous of degree less or equal than 2 (resp. 1, 0), and  $D_0$  is differential operator such that  $D_0\Gamma_0 \in C^m(V)$ .

Let now  $\phi \in C_c^\infty(V)$ .

We say that  $T_{0,0} \equiv T_{\xi_0,0}$  is a frozen operator at  $\xi_0$  of Type 0 if there exists a bounded measurable function  $\alpha_0 \equiv \alpha_{\xi_0}$  such that, for any  $\xi \in V$ ,

$$(T_{0,0}\phi)(\xi) = \text{P.V.} \int_V K_{0,0}(\xi, \eta)\phi(\eta) d\eta + \alpha_0(\xi)\phi(\xi);$$



We say then that  $T_{0,1} \equiv T_{\xi_0,1}$  (resp.  $T_{0,2} \equiv T_{\xi_0,2}$ ) is a frozen operator at  $\xi_0$  of Type 1 (resp. 2) if

$$(T_{0,1}\phi)(\xi) = \int_V K_{0,1}(\xi, \eta)\phi(\eta)d\eta \quad \left( \text{resp. } (T_{0,2}\phi)(\xi) = \int_V K_{0,2}(\xi, \eta)\phi(\eta)d\eta \right).$$

If, for any  $k = 0, 1, 2$ ,  $K_{0,k}(\xi, \eta) \equiv K_{\xi_0,k}(\xi, \eta)$  is a frozen kernel at  $\xi_0$  of type  $k$ , then we say that  $K_k(\xi, \eta) \equiv K_{\xi,k}(\xi, \eta)$  is a Kernel of Type  $k$ .

Finally we say that  $T_0 \equiv T_{\xi,0}$  is an Operator of Type 0 if there exists a bounded measurable function  $\alpha_0(\xi) \equiv \alpha_{\xi_0}(\xi)$  such that, setting  $\alpha(\xi) \equiv \alpha_{\xi}(\xi)$ , it results

$$(T_0\phi)(\xi) = \text{P.V.} \int_V K_0(\xi, \eta)\phi(\eta)d\eta + \alpha(\xi)\phi(\xi);$$

analogously we say that  $T_1 \equiv T_{\xi,1}$  (resp.  $T_2 \equiv T_{\xi,2}$ ) is an Operator of Type 1 (resp. 2) if

$$(T_1\phi)(\xi) = \int_V K_1(\xi, \eta)\phi(\eta)d\eta \quad \left( \text{resp. } (T_2\phi)(\xi) = \int_V K_2(\xi, \eta)\phi(\eta)d\eta \right).$$

According with above notations, the following facts hold.

**Lemma 1.** If, for any  $k = 1, 2$ ,  $K_{0,k}(\xi, \eta)$  is a frozen kernel at  $\xi_0$  of type  $k$ , then  $(\widetilde{X}_i K_{0,k}(\cdot, \eta))(\xi)$  is a frozen kernel at  $\xi_0$  of type  $k - 1$ .

If, for any  $k = 1, 2$ ,  $T_{0,k}$  is a frozen kernel at  $\xi_0$  of type  $k$ , then  $\widetilde{X}_i T_{0,k}$  is a frozen operator at  $\xi_0$  of type  $k - 1$ .  $\square$

**Example 1.** We recall that, for instance, fixed  $\xi_0 \in V$ , if  $i, j = 1, \dots, q$ , then

- $a(\xi)\Gamma_0(\Theta(\eta, \xi))b(\eta)$ ,
- $a(\xi)(R_i^\eta \Gamma_0)(\Theta(\eta, \xi))b(\eta)$ ,

are frozen kernel at  $\xi_0$  of type 2; while,

- $a(\xi)(Y_i \Gamma_0)(\Theta(\eta, \xi))b(\eta)$ ,
- $a(\xi)(Y_i R_j^\eta \Gamma_0)(\Theta(\eta, \xi))b(\eta)$ ,
- $a(\xi)(R_i^\eta R_j^\eta \Gamma_0)(\Theta(\eta, \xi))b(\eta)$ ,

are frozen kernel at  $\xi_0$  of type 1.

Let us conclude the present section with the following theorems whose proofs is either taken or adapted from the ones in [Brm-Brn2]).

**Theorem 6.** Let  $T_{0,k}$  be a frozen operator at  $\xi_0 \in V$  of type  $k = 0, 1, 2$ . Then, for any vector field  $\widetilde{X}_i$  there exist  $q + 1$  operators  $T_{0,k}^0, T_{0,k}^1, \dots, T_{0,k}^q$  frozen at  $\xi_0$ , of the same type  $k$  of  $T_{0,k}$  such that

$$\widetilde{X}_i T_{0,k} = \sum_{h=1}^q T_{0,k}^h \widetilde{X}_h + T_{0,k}^0$$

**Theorem 7.** Let  $T_0$  an operator of type 0 and  $1 < p < \infty$ . Then there exists a constant  $c \equiv c(T_0, p)$  such that for any  $u \in L^p(V)$  and for any  $a \in BMO_{\widetilde{X}}(V)$  it results

- (1)  $\|T_0 u\|_{L^p(V)} \leq c \|u\|_{L^p(V)}$ ;
- (2)  $\|[T_0, a](u)\|_{L^p(V)} \leq c \|a\|_{BMO_{\widetilde{X}}(V)} \|u\|_{L^p(V)}$ ;
- (3) if moreover  $a \in VMO_{\widetilde{X}}(V)$  and  $\epsilon > 0$ , then there exists  $r > 0$  depending on  $p, T_0, \epsilon$  and  $a$  such that, for any ball  $\widetilde{B}$  associated to the metric induced from the vector fields  $\widetilde{X}_1, \dots, \widetilde{X}_q$ , if  $\text{supp } u \subset \widetilde{B}$  then

$$\|[T_0, a](u)\|_{L^p(V)} \leq \epsilon \|u\|_{L^p(V)},$$

where  $BMO_{\widetilde{X}}(V)$  and  $VMO_{\widetilde{X}}(V)$  denote function spaces as in Section (2.6), with respect to  $C$ - $C$  metric induced from vector fields  $\widetilde{X}_1, \dots, \widetilde{X}_q$ , and  $[T_0, a]$  denotes the commutator which maps  $u \in L^p(V)$  into  $T(au) - aT(u)$ .  $\square$

**Theorem 8.** Let  $T_k$  be an operator of type  $k = 1, 2$  and  $1 < p < \frac{Q}{k}$ . Then there exists a constant  $c \equiv c(T_k, p)$  such that, if  $\frac{1}{q} = \frac{1}{p} + \frac{k}{Q}$ , then, for any  $u \in L^p(V)$  it results

$$\|T_k u\|_{L^q(V)} \leq c \|u\|_{L^p(V)}. \quad \square$$

**Notation 2.** Set  $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{Q}$  and, for  $p < Q$ ,  $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}$  (see Step 4 pag. (3)). It is easy verified that  $p_* < p < p^*$ ,  $(p_*)^* = (p^*)_* = p$  and, moreover, the mapping that associate to  $p$  any one of the corresponding “star  $p$ ” is order preserving in the real numbers; for instance, from  $p_* < p < p^*$  it follows that  $p < p^* < p^{**}$  then  $p^* < p^{**} < p^{***}$ , and so on

**Corollary 1.** Let  $T_k$  un operator of type  $k = 1, 2$  and  $1 < p < \frac{Q}{2}$ . Then there exists a constant  $c \equiv c(T_1, T_2, p)$  and  $V$  such that for any  $u \in L^p(V)$  it results

$$\begin{aligned} \|T_1 u\|_{L^p(V)} &\leq c \|u\|_{L^{p_*}(V)} \\ \|T_2 u\|_{L^p(V)} &\leq c \|u\|_{L^{p^*}(V)}. \quad \square \end{aligned}$$

**2.5. Sobolev spaces associated to a family of Lipschitz vector fields.** Let  $\Omega \subset \mathbb{R}^n$  an open set and  $Y : \Omega \rightarrow \mathbb{R}^n$  a Lipschitz vector field

$$Y(x) = \sum_{i=1}^n b_i(x) \partial_i \equiv (b_1(x), \dots, b_n(x)) \quad \forall x \in \Omega.$$

Assuming, for instance, Lipschitz regularity for  $\partial\Omega$ , if  $f, b_i \in C^1(\overline{\Omega})$ ,  $i = 1, \dots, n$ , and  $\phi \in C_c^\infty(\Omega)$  then

$$\int_{\Omega} Y f \phi \, dx = \sum_{i=1}^n \left[ \int_{\Omega} \partial_i (b_i f \phi) \, dx - \int_{\Omega} f \partial_i (b_i \phi) \, dx \right] = \int_{\Omega} f \left( - \sum_{i=1}^n \partial_i (b_i \phi) \right) \, dx.$$

If we set  $Y^T = - \sum_{i=1}^n \partial_i (b_i \cdot)$  then we can write

$$\int_{\Omega} Y f \phi \, dx = \int_{\Omega} f Y^T \phi \, dx$$

so it suffices to request for any  $b_i$  to be locally Lipschitz. If  $Y = \sum_{i=1}^n b_i \partial_i$  is a locally Lipschitz vector fields on  $\Omega$  and  $f, g \in L_{loc}^1(\Omega)$ , we say that  $g$  is the partial derivative along  $Y$ , and we write  $Y f = g$ , if

$$\int_{\Omega} g \phi \, dx = \int_{\Omega} f Y^T \phi \, dx$$

for any  $\phi \in C_c^\infty(\Omega)$ . Let  $X = (X_1, \dots, X_q)$  a family of locally Lipschitz vector fields on  $\Omega$ . if  $f \in L_{loc}^1(\Omega)$  has partial derivatives along  $X_j \, \forall j = 1, \dots, q$ , let us denote by  $X f = (X_1 f, \dots, X_q f)$  the *weak gradient* of  $f$ . Moreover we set  $|X f| = (|X_1 f|^2 + \dots + |X_q f|^2)^{\frac{1}{2}}$ .

Let  $1 \leq p < \infty$ , and let  $\{X_1, \dots, X_q\}$  be a family of locally Lipschitz vector fields on  $\Omega$ .

**Definition 2.** The Sobolev space  $W_X^{1,p}(\Omega)$  is the space of all function  $f : \Omega \rightarrow \mathbb{R}$  such that  $f \in L^p(\Omega)$  and, for any  $j = 1, \dots, q$ ,  $X_j f$  do exists in the weak sense and belong to  $L^p(\Omega)$ .

$W_X^{1,p}(\Omega)$  is a Banach space with the norm

$$\|f\|_{W_X^{1,p}(\Omega)} = \left( \|f\|_{L^p(\Omega)}^p + \sum_{j=1}^q \|X_j f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

for any  $1 \leq p < \infty$ , or equivalently with the norm  $\|f\|_{L^p(\Omega)} + \sum_{j=1}^q \|X_j f\|_{L^p(\Omega)}$ . Analogous definitions hold for the local Sobolev spaces  $W_{loc,X}^{1,p}(\Omega)$ , and for the subspace of all functions zero on  $\partial\Omega$ . Let us denote by  $\overset{\circ}{W}_X^{1,p}(\Omega)$  the closure of  $C_c^\infty(\Omega)$  in  $W_X^{1,p}(\Omega)$ . The following proposition recall basic properties of Sobolev spaces.

**Proposition 5.** *Let  $f, g \in W_X^{1,p}(\Omega)$ . Then*

- (i) *for any  $\lambda, \mu \in \mathbb{R}$ ,  $\lambda f + \mu g \in W_X^{1,p}(\Omega)$  e  $X_j(\lambda f + \mu g) = \lambda X_j f + \mu X_j g \quad \forall j = 1, \dots, q$ ;*
- (ii) *If  $U$  is an open subset of  $\Omega$  then  $f \in W_X^{1,p}(U)$ ;*
- (iii) *If  $\zeta \in C_c^\infty(\Omega)$  then  $\zeta f \in \overset{\circ}{W}_X^{1,p}(\Omega)$  e  $X_j(\zeta f) = \zeta X_j f + f X_j \zeta \quad \forall j = 1, \dots, q$ .*

Let us conclude this section recalling that for  $p = 2$ ,  $W_X^{1,2}(\Omega) \equiv H_X^1(\Omega)$  and  $\overset{\circ}{W}_X^{1,2}(\Omega) \equiv \overset{\circ}{H}_X^1(\Omega)$  have a natural structure of Hilbert space. In this setting, the weak formulation of Dirichlet problem

$$\begin{cases} X_j^T(a^{ij}X_i) = f \\ u \in \overset{\circ}{H}_X^1(\Omega) \end{cases}$$

with  $f \in H_X^1(\Omega)$ ,  $a^{ij}$  are bounded and measurable functions, and by assuming uniform ellipticity, has a unique solution: the proof follows standard Lax–Milgram lemma; analogously, classic  $L^2$  regularity theory holds.

**2.6. Space of homogeneous type: the space  $BMO_X$  and  $VMO_X$ .** For these definitions and results we refer the reader to the whole paper [CarFan].

### 3. REGULARITY RESULT

In this section we give a detailed sketch of the proof of Theorem (1) of pag. 1, postponing at the end main calculations.

*Sketch of the proof.* According to notation of Theorem (2), let  $W = B \times I$  where  $B = B(x, r)$  is an Euclidean open ball,  $x \in \Omega'$ ,  $I$  is an open rectangle in  $\mathbb{R}^k$  centered in zero, and  $r > 0$  is sufficiently small. It suffices to verify that for any  $\xi \in W$  there exist two open  $C$ – $C$  balls  $B'_X \Subset B''_X \Subset W$  around  $\xi$  such that

$$(3.1) \quad \|\tilde{u}\|_{W_X^{1,p}(B'_X)} \leq c \left( \|\tilde{u}\|_{L^p(B''_X)} + \|\tilde{F}\|_{L^p(B''_X, \mathbb{R}^q)} \right).$$

(Step 1) Let  $p \geq 2$  fixed. The function  $u \in W_X^{1,p}(\Omega)$  is a weak solution in  $\Omega$  of the equation (1.1); then  $\tilde{u} \in W_X^{1,p}(W)$  is a weak solution in  $W$  of the equation

$$(3.2) \quad \widetilde{X}_j^T(a^{ij}\widetilde{X}_i\tilde{u}) = \widetilde{X}_j^T\tilde{F}^j + f,$$

where  $\widetilde{X}_1, \dots, \widetilde{X}_q$  are the free vector fields as in Theorem (2),

$$(3.3) \quad f = (a^{ij}\widetilde{X}_i\tilde{u} - \tilde{F}^j)g_j,$$

and  $g_j$  are smooth functions in  $W$ , defined through the positions

$$(3.4) \quad g_j = - \left( \partial_j \lambda_{jN+1} + \dots + \partial_j \lambda_{jN} \right) = - \operatorname{div}_{\text{Euclidean}}(\lambda_{jN+1}, \dots, \lambda_{jN}), \quad j = 1, \dots, q.$$

(Step 2) First we suppose that  $\tilde{u} \in C_c^\infty(W)$  is a weak solution of the equation

$$(3.5) \quad \widetilde{X}_j^T(a^{ij}\widetilde{X}_i\tilde{u}) = \widetilde{X}_j^T\tilde{F}^j + g,$$

where  $g \in C_c^\infty(W)$  is fixed,  $\tilde{F} \in C_c^\infty(W, \mathbb{R}^q)$ ,  $\tilde{a}^{ij} \in C^\infty(W) \cap VMO_{\tilde{X}}(W)$ ; we prove that if  $\text{supp } u \subset B_{\tilde{X}}$ , with  $B_{\tilde{X}}$  open  $C$ - $C$  ball with radius  $\sigma < \bar{\sigma}$ ,  $\bar{\sigma}$  sufficiently small, than hypothesis  $\tilde{a}^{ij} \in VMO_{\tilde{X}}(W)$  yields

$$(3.6) \quad \|\tilde{X}\tilde{u}\|_{L^p(B_{\tilde{X}})} \leq c \left( \|\tilde{u}\|_{L^p(B_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^{p*}(B_{\tilde{X}})} + \|g\|_{L^{p*}(B_{\tilde{X}})} \right),$$

where  $p_*$  is as in Notation (2), Section (2.4).

This step requires more care than the following ones; indeed we have to prove the existence of a parametrix, more precisely of an operator that, up to a finite sum of operators of type 0, 1 and 2, behaves like a right inverse and, actually, also as a left inverse because of the symmetry of the matrix with entries  $\tilde{a}^{ij}$ . The construction of this parametrix, as shown in Section (2.4), makes use of results in Section “PART III. OPERATORS CORRESPONDING TO FREE VECTOR FIELDS” in [RS], and Section “2. DIFFERENTIAL OPERATORS AND FUNDAMENTAL SOLUTIONS” in [Brm-Brn2]; in particular the estimate is obtained through results of Section (2.4).

(Step 3) Now we suppose that  $\tilde{u} \in C^\infty(W)$ ,  $\tilde{F} \in C^\infty(W, \mathbb{R}^q)$ ,  $\tilde{a}^{ij} \in C^\infty(W)$ ,  $g \in C^\infty(W)$  is fixed and  $\tilde{u}$  is a solution of (3.5). Fix  $\xi \in W$ ,  $0 < \gamma < 1$  and  $0 < \sigma < \bar{\sigma}$ . Arguing as in Lemma 3.3 in [Brm-Brn2] we choose  $\theta \in C_c^\infty(W)$  such that  $B'_{\tilde{X}} \prec \theta \prec B''_{\tilde{X}}$ , where  $B'_{\tilde{X}} \Subset B''_{\tilde{X}} \Subset W$  are concentric balls around  $\xi$ , with respective radii  $\gamma\sigma < \sigma$ , and such that  $|\tilde{X}\theta| \leq \frac{c}{(1-\gamma)\sigma}$ . Then the function  $\theta u \in C_c^\infty(W)$  is a weak solution of

$$(3.7) \quad \tilde{X}_j^T (\tilde{a}^{ij} \tilde{X}_i (\theta \tilde{u})) = \tilde{X}_j^T (\tilde{a}^{ij} (\tilde{u} \tilde{X}_i \theta + \theta \tilde{F}^j)) + (g\theta - \tilde{a}^{ij} (\tilde{X}_i \tilde{u}) (\tilde{X}_j \theta))$$

and, by applying previous Step 2, the following estimate holds

$$(3.8) \quad \|\tilde{X}\tilde{u}\|_{L^p(B'_{\tilde{X}})} \leq c \left( \|\tilde{u}\|_{L^p(B''_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^{p*}(B''_{\tilde{X}})} + \|g\|_{L^{p*}(B''_{\tilde{X}})} \right).$$

(Step 4) Fixed  $2 < p \leq 2^*$ , let us suppose that  $\tilde{u} \in W_{\tilde{X}}^{1,p}(W)$  is a solution of (3.2). Let  $\tilde{a}^{ij}_h \in C^\infty(W) \cap VMO_{\tilde{X}}(W)$   $q^2$  function's sequences converging respectively to  $\tilde{a}^{ij}$ , let  $\tilde{F}_h \in C^\infty(W) \cap L^p(W, \mathbb{R}^q)$  a function's sequence converging in  $L^p(W, \mathbb{R}^q)$  to  $\tilde{F}$ , and let  $f_h \in C^\infty(W) \cap L^p(W)$  a function's sequence converging in  $L^p(W)$  to  $f$ . Let us consider the sequence of Dirichlet problems

$$(3.9) \quad (D_h) \quad \begin{cases} \tilde{X}_j^T (\tilde{a}^{ij}_h (\tilde{X}_i \tilde{u}_h)) = \tilde{X}_j^T \tilde{F}_h^j + f_h \\ \tilde{u}_h - \tilde{u} \in \dot{H}_{\tilde{X}}^1(W) \end{cases}.$$

For any  $h = 1, 2, \dots$ , let  $\tilde{u}_h \in H_{\tilde{X}}^1(W)$  the unique weak solution of  $(D_h)$ . Then, for any  $h = 1, 2, \dots$ , the function  $v_h = \tilde{u}_h - \tilde{u}$  is solution of the problem

$$(3.10) \quad (D'_h) \quad \begin{cases} \tilde{X}_j^T (\tilde{a}^{ij}_h (\tilde{X}_i v_h)) = \tilde{X}_j^T (\tilde{F}_h^j - \tilde{F}) + (f_h - f) - \tilde{X}_j^T ((\tilde{a}^{ij}_h - \tilde{a}^{ij}) \tilde{X}_i \tilde{u}) \\ v_h \in \dot{H}_{\tilde{X}}^1(W) \end{cases}.$$

So, for any  $h = 1, 2, \dots$ , we have

$$(3.11) \quad \|v_h\|_{H_{\tilde{X}}^1(W)} \leq c \left( \|\tilde{F}_h - \tilde{F}\|_{L^2(W, \mathbb{R}^q)} + \|f_h - f\|_{L^2(W)} + \|(\tilde{a}^{ij}_h - \tilde{a}^{ij}) \tilde{X}_i \tilde{u}\|_{L^2(W)} \right),$$

from which it follows that  $\tilde{u}_h \rightarrow \tilde{u}$  in  $H_{\tilde{X}}^1(W)$ . Let us observe now that  $\tilde{u}_h \in C^\infty(W)$ . Indeed the operators  $\tilde{X}_j (\tilde{a}^{ij}_h \tilde{X}_i)$  have smooth coefficients in

$W$ , then can be equivalently be written as a sum of an operator in non divergence form and one term of less degree, that is of the kind  $-\widetilde{a}^{ij}_h \widetilde{X}_j \widetilde{X}_i + b^i_h \widetilde{X}_i$ , with  $b^i \in C^\infty(W)$ . From the properties of convolution, and arguing as in Teorema 1.11 di [Brm-Brn1], we obtain that any  $\widetilde{X}_j(\widetilde{a}^{ij}_h \widetilde{X}_i)$  is hypoelliptic. It follows that  $\widetilde{u}_h \in W_X^{1,p}(W)$  for any  $1 \leq p \leq \infty$ . So we can apply to each function  $\widetilde{u}_h$  local De Giorgi–Stampacchia–Moser estimates; more precisely, when  $p \geq 2$ , because  $\widetilde{u}_h \in H_X^1(W)$  is solution of the equation in  $(D_h)$ , it follows that, up to a shrinking of  $W$ , there exists  $0 < \tilde{\sigma} < \bar{\sigma}$ , such that, for any  $\xi \in W$ , we can find two  $C$ – $C$  open balls, say  $\overline{B}'_{\tilde{X}}$  e  $\overline{B}''_{\tilde{X}}$ , of suitable radii  $0 < \rho' < \rho'' < \tilde{\sigma}$ , not depending on the given point, and a constant  $c = c(p, \rho', \rho'', \nu, Q)$  such that  $\|\widetilde{u}_h\|_{L^\infty(\overline{B}'_{\tilde{X}})} \leq c(\|\widetilde{u}_h\|_{L^2(\overline{B}''_{\tilde{X}})} + 1)$ : last formula implies that  $\widetilde{u}_h$  are uniformly bounded in  $L^p(\overline{B}'_{\tilde{X}})$ . Applying now *Step 3*, for any point  $\xi \in W$  we find two  $C$ – $C$  open balls, say  $B'_{\tilde{X}} \Subset B''_{\tilde{X}}$ , with radii  $0 < \gamma\sigma < \sigma < \tilde{\sigma}$  such that

(3.12)

$$\|\widetilde{X}\widetilde{u}_h\|_{L^p(B'_{\tilde{X}})} \leq c\left(\|\widetilde{u}_h\|_{L^p(B''_{\tilde{X}})} + \|\widetilde{F}_h\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\widetilde{X}\widetilde{u}_h\|_{L^{p_*}(B''_{\tilde{X}})} + \|f_h\|_{L^{p_*}(B''_{\tilde{X}})}\right)$$

and, moreover,  $u_h$  are uniformly bounded in the biggest ball. This fact, considering that  $p_* < 2$  and that  $\{\widetilde{u}_h\}$  is bounded in  $H_X^1(W)$  implies  $\|\widetilde{u}_h\|_{W_X^{1,p}(B'_{\tilde{X}})} \leq \text{constant}$  for any  $h = 1, 2, \dots$ . So there exists a subsequence, that we keep calling  $\{\widetilde{u}_h\}$ , which weakly converges in  $W_X^{1,p}(B'_{\tilde{X}})$  to a given  $\bar{u} \in W_X^{1,p}(B'_{\tilde{X}})$ . But from  $\widetilde{u}_h \xrightarrow{W_X^{1,2}(B'_{\tilde{X}})} \bar{u}$  it follows that  $\widetilde{u}_h \xrightarrow{W_X^{1,p}(B'_{\tilde{X}})} \bar{u}$ , so  $\bar{u} = \bar{u}$  in  $B'_{\tilde{X}}$ . By the uniform boundedness principle finally it follows that, for any  $2 < p \leq 2^*$ ,

$$\begin{aligned} \|\widetilde{X}\bar{u}\|_{L^p(B'_{\tilde{X}})} &\leq c\left(\|\bar{u}\|_{L^p(B''_{\tilde{X}})} + \|\widetilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\widetilde{X}\bar{u}\|_{L^{p_*}(B''_{\tilde{X}})} + \|f\|_{L^{p_*}(B''_{\tilde{X}})}\right) \\ &\leq c\left(\|\bar{u}\|_{L^p(B''_{\tilde{X}})} + \|\widetilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\widetilde{X}\bar{u}\|_{L^{p_*}(B''_{\tilde{X}})}\right) \\ &\leq c\left(\|\bar{u}\|_{L^p(B'_{\tilde{X}})} + \|\widetilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\widetilde{X}\bar{u}\|_{L^2(B''_{\tilde{X}})}\right). \end{aligned}$$

(3.13)

(*Step 5*) Here we employ the recursive technique as in [DF]. More precisely, by applying repetitively *Step 3* and *Step 4*, we will show the existence of  $L^p$  estimate when the number  $Q$  “grows up” with  $p$ . More precisely we have  $]2, \infty[ = ]2, 2^*] \cup ]2^*, 2^{**}] \cup \dots$  where, for  $2 \leq q < Q$ ,  $q^*$  is defined as in Notation (2). In particular, for  $q = 2$ ,  $q^{**}$  is defined only when  $Q > 4$ , and so on. Let us suppose that  $\bar{u} \in W_X^{1,p}(W)$  is a solution of (3.2). Then *Step 4* implies that if  $2 < p \leq 2^*$  and  $0 < \sigma < \bar{\sigma}$  is sufficiently small, for any  $\xi \in W$ , and relatively to balls  $B'_{\tilde{X}} \Subset B''_{\tilde{X}}$  around  $\xi \in W$  with radii  $0 < \gamma\sigma < \sigma$ , (3.13) holds. Fixed now  $2^* < p \leq 2^{**}$  and repeat again arguments in *Step 3* and *Step 4*. Choose a function  $\theta \in C_c^\infty(B'_{\tilde{X}})$ , where  $B'_{\tilde{X}}$  is, for any fixed  $\xi$ , the open  $C$ – $C$  ball around  $\xi$  with radius  $\gamma\sigma$ ; more precisely we choose this ball such that  $\overline{B}_{\tilde{X}} \prec \theta \prec B'_{\tilde{X}}$ , where  $\overline{B}_{\tilde{X}} \Subset B'_{\tilde{X}}$  are two open  $C$ – $C$  concentric balls around  $\xi$ , with radii  $0 < \gamma^2\sigma < \gamma\sigma$ , and such that  $|\widetilde{X}\theta| \leq \frac{c}{(1-\gamma)\gamma\sigma}$ .

Then

$$\begin{aligned}
\|\tilde{X}\tilde{u}\|_{L^p(\overline{B}_{\tilde{X}})} &\leq c\left(\|\tilde{u}\|_{L^p(B'_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B'_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^{p*}(B'_{\tilde{X}})}\right) \\
&\leq c\left(\|\tilde{u}\|_{L^p(B'_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B'_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^{2*}(B'_{\tilde{X}})}\right), \\
&\leq c\left(\|\tilde{u}\|_{L^p(B''_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{u}\|_{L^{2*}(B''_{\tilde{X}})} + \|\tilde{F}\|_{L^{2*}(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^2(B''_{\tilde{X}})}\right) \\
(3.14) \quad &\leq c\left(\|\tilde{u}\|_{L^p(B''_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^2(B''_{\tilde{X}})}\right).
\end{aligned}$$

So, if  $p > 2$  let  $h \in \mathbb{N}$  be such that  $2^{\overbrace{* \dots *}^{h-1}} < p \leq 2^{\overbrace{* \dots *}^h}$ . If  $\gamma = (\frac{1}{2})^{\frac{1}{h}}$ , by reitering the argument we obtain

$$(3.15) \quad \|\tilde{X}\tilde{u}\|_{L^p(B'_{\tilde{X}})} \leq c\left(\|\tilde{u}\|_{L^p(B''_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B''_{\tilde{X}}, \mathbb{R}^q)} + \|\tilde{X}\tilde{u}\|_{L^2(B''_{\tilde{X}})}\right)$$

where  $B'_{\tilde{X}} \Subset B''_{\tilde{X}}$  are two open  $C$ - $C$  balls of radii  $0 < \frac{\sigma}{2} < \sigma$ , for any  $0 < \sigma < \tilde{\sigma}$ .

(Step 6) So far we have obtained that if  $2 \leq p < \infty$  and  $\tilde{u} \in W_{\tilde{X}}^{1,p}(W)$  is a solution of (3.2), then the estimate (3.15) holds. Observe now that in particular  $\tilde{u} \in W_{\tilde{X}}^{1,2}(W)$  is a solution of (3.2): then classical  $L^2$  theory yields the existence of  $0 < \rho < \tilde{\sigma}$  such that for any  $2\sigma < \rho$ , if  $B'''_{\tilde{X}}$  is the open ball of radius  $2\sigma$  concentric with  $B''_{\tilde{X}}$ , it results

$$\begin{aligned}
\|\tilde{X}\tilde{u}\|_{L^2(B'''_{\tilde{X}})} &\leq c\left(\|\tilde{u}\|_{L^2(B'''_{\tilde{X}})} + \|\tilde{F}\|_{L^2(B'''_{\tilde{X}}, \mathbb{R}^q)}\right) \\
&\leq c\left(\|\tilde{u}\|_{L^p(B'''_{\tilde{X}})} + \|\tilde{F}\|_{L^p(B'''_{\tilde{X}}, \mathbb{R}^q)}\right)
\end{aligned}$$

because  $2 < p$ . This last estimate, jointly with (3.15) implies (3.1).

(Step 7) Noting now that  $\sigma$  is as small as we need, in (3.1), taking into account the local equivalence of  $d_{\tilde{X}}$  and the Euclidean metric we can assume that (3.1) holds with two Euclidean balls; then the estimate clearly holds between due open sets of the kind  $B \times I, B' \times I$ , with  $B \Subset B' \subseteq \Omega$  open Euclidean balls small enough and  $I$  open rectangle in  $\mathbb{R}^k$ , previously fixed small enogh. Then through Proposition 1.4 of [FSSC] (applied with the weight function  $w \equiv 1$ ) and Remark (1), we obtain the existence of an absolute constant such that, for any  $u \in W_X^{1,p}(W)$  solution of (1.1), the following estimate holds

$$(3.16) \quad \|u\|_{W_X^{1,p}(B)} \leq c\left(\|u\|_{L^p(B')} + \|F\|_{L^p(B', \mathbb{R}^q)}\right),$$

and so the thesis.

Finally let us sketch some of the proofs.

(Step 1 - Some Remarks) If  $u \in S_X^{1,p}(\Omega)$ , then  $\tilde{u} \in W_{\tilde{X}}^{1,p}(W)$ . Indeed, by Proposition 1.4 in [FSSC], we can assume that  $\tilde{u} \in C^\infty(\tilde{W})$ . Then  $\mathcal{D}(B) \otimes \mathcal{D}(I)$  is dense into  $\mathcal{D}(W)$  and, in particular, we can consider test functions of the kind  $\Phi(\xi) \equiv \phi(x)\psi(t)$  for any  $\xi = (x, t) \in B \times I$ , with arbitrary  $\phi \in \mathcal{D}(B)$  and  $\psi \in \mathcal{D}(I)$ . So, in (1.2), it suffices to multiply by  $\psi(t)$  and integrate over  $I$ , in order to apply Remark (1). In particular  $\tilde{X}_j(u \circ \pi) = X_j u \circ \pi$  holds for any  $u \in S_X^{1,p}(\Omega)$ . Analogous arguments hold for (3.2).



(Step 2 - Proof) We refer to notations and results in Section (2.4). Fix  $\xi_0 \in W$  and let us consider the operator  $\tilde{L}_0 \equiv \tilde{L}_{\xi_0} = \tilde{X}_j^T (\tilde{a}^{ij}(\xi_0) \tilde{X}_i)$ . We have to construct two partial inverse  $D_0$  and  $S_0$  of  $\tilde{L}_0$  of type 2, frozen at  $\xi_0$ . We have  $\tilde{L} = \tilde{L}_0 + (\tilde{L} - \tilde{L}_0)$ ; then we can apply to both member of the equation a suitable operator of type 1 and, finally, after some calculations, we can free  $\xi_0$  having so a representation of  $\tilde{X}_i u$ , so to apply Theorems (7) and (1) to obtain (3.6). Precisely, let us fix  $a \in C_c^\infty(W)$  and  $\xi_0 \in W$ . We are going to prove that there exist two frozen operators at  $\xi_0$ ,  $D_{0,2}$  and  $S_{0,2}$  of type 2, and a finite number of operators frozen at  $\xi_0$  (depending only on  $X_j^T(a^{ij} X_i)$ )  $D_{0,1}^{ij,h}$  e  $D_{0,2}^{ij,k}$ ,  $h \in I$ ,  $k \in J$  with  $|I|, |J| \leq$  absolute constant, which are Riesz potential of type 1 and 2, such that, for any test  $\tilde{u}$  in  $W$  it results

$$(3.17) \quad \tilde{L}_0 D_{0,2} \tilde{u} = -au + \sum_{h \in I} \sum_{i,j=1}^q \tilde{a}^{ij}(\xi_0) D_{0,1}^{ij,h} \tilde{u} + \sum_{k \in K} \sum_{i,j=1}^q \tilde{a}^{ij}(\xi_0) D_{0,2}^{ij,k} \tilde{u},$$

$$(3.18) \quad S_{0,2} \tilde{L}_0 \tilde{u} = -au + \sum_{h \in I} \sum_{i,j=1}^q \tilde{a}^{ij}(\xi_0) S_{0,1}^{ij,h} \tilde{u} + \sum_{k \in K} \sum_{i,j=1}^q \tilde{a}^{ij}(\xi_0) S_{0,2}^{ij,k} \tilde{u}.$$

To this aim, let us consider the fundamental solution  $\Gamma_0$  ensured by Folland's theorem, of the invariant operator  $\tilde{a}^{ji}(\xi_0) Y_j Y_i$  associated to the non divergence form operator  $\tilde{a}^{ji}(\xi_0) \tilde{X}_j \tilde{X}_i$  (see Theorem (5)). Let us consider now the operator that in [Brm-Brn2] is adapted to the one of [RS]; more precisely, fixed a test  $b$  in  $W$  such that  $\text{supp } a \subset \{b = 1\}$ , according to notations of Theorem (4), we set, for any  $\tilde{u}$  test in  $W$ , and for any  $\xi \in W$ ,

$$(3.19) \quad D_{0,2} \tilde{u}(\xi) = \frac{a(\xi)}{\omega(\xi)} \int_W \Gamma(\Theta(\eta, \xi)) b(\eta) \tilde{u}(\eta) d\eta.$$

Applying Theorem (3), for any  $i = 1, \dots, q$  we have

$$(3.20) \quad \begin{aligned} \tilde{X}_i D_{0,2} \tilde{u}(\xi) &= \tilde{X}_i \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \Gamma(\Theta(\eta, \xi)) b(\eta) \tilde{u}(\eta) d\eta \\ &+ \frac{a(\xi)}{\omega(\xi)} \int_W \left[ Y_i \Gamma_0(\Theta(\eta, \xi)) + R_i^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \tilde{u}(\eta) d\eta. \end{aligned}$$

Now, according to Theorem (2), if  $c_j = (a_{j1}, \dots, a_{jn}, \lambda_{jn+1}, \dots, \lambda_{jN})$  denote entries of the vector field  $\tilde{X}_j$ , it is  $\tilde{X}_j^T = -\tilde{X}_j + m_j$  where  $m_j(\xi) = (\text{div}_{\text{euclidean}} c_j)(\xi)$ , for any  $\xi \in W$ . So, arguing as in [Brm-Brn2], we have

$$\begin{aligned} \tilde{L}_0 D_{0,2} \tilde{u}(\xi) &= -\tilde{X}_j \left[ \tilde{a}^{ji}(\xi_0) \tilde{X}_i \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \Gamma_0(\Theta(\eta, \xi)) b(\eta) \tilde{u}(\eta) d\eta \right. \\ &\quad \left. + \tilde{a}^{ji}(\xi_0) \frac{a(\xi)}{\omega(\xi)} \int_W \left[ Y_i \Gamma_0(\Theta(\eta, \xi)) + R_i^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \tilde{u}(\eta) d\eta \right] \\ &+ m_j(\xi) \left[ \tilde{a}^{ji}(\xi_0) \tilde{X}_i \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \Gamma_0(\Theta(\eta, \xi)) b(\eta) \tilde{u}(\eta) d\eta \right. \\ &\quad \left. + \tilde{a}^{ji}(\xi_0) \frac{a(\xi)}{\omega(\xi)} \int_W \left[ Y_i \Gamma_0(\Theta(\eta, \xi)) + R_i^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \tilde{u}(\eta) d\eta \right] = \end{aligned}$$

$$\begin{aligned}
&= \widetilde{a}^{ji}(\xi_0) \left[ \widetilde{X}_j \widetilde{X}_i \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \Gamma_0(\Theta(\eta, \xi)) b(\eta) \widetilde{u}(\eta) d\eta \right. \\
&\quad + \widetilde{X}_i \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \left[ Y_j \Gamma_0(\Theta(\eta, \xi)) + R_j^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \widetilde{u}(\eta) d\eta \\
&\quad + \widetilde{X}_j \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \left[ Y_i \Gamma_0(\Theta(\eta, \xi)) + R_i^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \widetilde{u}(\eta) d\eta \\
&\quad + \frac{a(\xi)}{\omega(\xi)} P.V. \int_W \left[ Y_j Y_i \Gamma_0(\Theta(\eta, \xi)) + Y_j R_i^\xi \Gamma_0(\Theta(\eta, \xi)) + \right. \\
&\quad \left. + R_j^\xi Y_i \Gamma_0(\Theta(\eta, \xi)) + R_j^\xi R_i^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \widetilde{u}(\eta) d\eta \Big] \\
&+ m_j(\xi) \left[ \widetilde{a}^{ji}(\xi_0) \widetilde{X}_i \left( \frac{a(\xi)}{\omega(\xi)} \right) \int_W \Gamma_0(\Theta(\eta, \xi)) b(\eta) \widetilde{u}(\eta) d\eta \right. \\
&\quad \left. + \widetilde{a}^{ji}(\xi_0) \frac{a(\xi)}{\omega(\xi)} \int_W \left[ Y_i \Gamma_0(\Theta(\eta, \xi)) + R_i^\xi \Gamma_0(\Theta(\eta, \xi)) \right] b(\eta) \widetilde{u}(\eta) d\eta \right] \\
(3.21) \quad &= -au + \sum_{h \in I} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) D_{0,1}^{ij,h} \widetilde{u} + \sum_{k \in K} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) D_{0,2}^{ij,k} \widetilde{u}.
\end{aligned}$$

Finally, by transposition of the matrix  $(\widetilde{a}^{ij})_{i,j=1,\dots,q}$  we obtain the desired formula. In particular, fixed any test  $\widetilde{u}$ , for any test function  $a$  such that  $\text{supp } \widetilde{u} \subset \{a = -1\}$ , we can write  $\widetilde{X}_m \widetilde{u}$ . More precisely, by applying  $\widetilde{X}_m$  to both member of (3.18), by Theorem (6), for any  $m = 1, \dots, q$  it is

$$\begin{aligned}
\widetilde{X}_m S_{0,2} \widetilde{L}_0 u &= \widetilde{X}_m u + \sum_{h \in I} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) \left[ \sum_{l=1}^q T_{0,1}^{ij,hl} \widetilde{X}_l + T_{0,1}^{ij,h} \right] \widetilde{u} \\
(3.22) \quad &+ \sum_{k \in K} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) \left[ \sum_{l=1}^q T_{0,2}^{ij,kl} \widetilde{X}_l + T_{0,2}^{ij,k} \right] \widetilde{u}.
\end{aligned}$$

It follows that, thanks to Lemma (1),  $T_{0,1} \equiv \widetilde{X}_m S_{0,2}$ , is an operator of type 1 frozen at  $\xi_0$ . So, let  $\widetilde{u}$  be a solution of (3.5). By applying  $T_{0,1}$  to

$$\begin{aligned}
\widetilde{L}_0 \widetilde{u} &= \widetilde{L} \widetilde{u} + (\widetilde{L}_0 - \widetilde{L}) \widetilde{u} \\
(3.23) \quad &= \widetilde{X}_j^T \widetilde{F}^j + g + \widetilde{X}_j^T ((\widetilde{a}^{ij}(\xi_0) - \widetilde{a}^{ij}) \widetilde{X}_i \widetilde{u})
\end{aligned}$$

it is

$$(3.24) \quad T_{0,1} \widetilde{L}_0 \widetilde{u} = T_{0,1} \widetilde{X}_j^T \widetilde{F}^j + T_{0,1} g + T_{0,1} \left[ \widetilde{X}_j^T ((\widetilde{a}^{ij}(\xi_0) - \widetilde{a}^{ij}) \widetilde{X}_i \widetilde{u}) \right].$$

Then, putting (3.24) into (3.22) and getting  $\widetilde{X}_m \widetilde{u}$ , we have

$$\begin{aligned}
\widetilde{X}_m \widetilde{u} &= T_{0,1} \widetilde{X}_j^T \widetilde{F}^j + T_{0,1} g \\
&+ T_{0,1} \left[ \widetilde{X}_j^T ((\widetilde{a}^{ij}(\xi_0) - \widetilde{a}^{ij}) \widetilde{X}_i \widetilde{u}) \right] \\
&- \sum_{h \in I} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) \left[ \sum_{l=1}^q T_{0,1}^{ij,hl} \widetilde{X}_l + T_{0,1}^{ij,h} \right] \widetilde{u} \\
(3.25) \quad &- \sum_{k \in K} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) \left[ \sum_{l=1}^q T_{0,2}^{ij,kl} \widetilde{X}_l + T_{0,2}^{ij,k} \right] \widetilde{u}.
\end{aligned}$$

Finally, through the definition of transposed operator and Lemma (1), it follows that for  $q$  suitable operators  $T_{0,0}^j$  frozen at  $\xi_0$  of type 0, (3.25) becomes

$$\begin{aligned}
 \widetilde{X}_{im} \widetilde{u} &= \sum_{j=1}^q T_{0,0}^j \widetilde{F}^j + T_{0,1} g \\
 &+ \sum_{j=1}^q T_{0,0}^j ((\widetilde{a}^{ij}(\xi_0) - \widetilde{a}^{ij}) \widetilde{X}_i \widetilde{u}) \\
 &- \sum_{h \in I} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) \left[ \sum_{l=1}^q T_{0,1}^{ij,hl} \widetilde{X}_l + T_{0,1}^{ij,l} \right] \widetilde{u} \\
 &- \sum_{k \in K} \sum_{i,j=1}^q \widetilde{a}^{ij}(\xi_0) \left[ \sum_{l=1}^q T_{0,2}^{ij,kl} \widetilde{X}_l + T_{0,2}^{ij,k} \right] \widetilde{u}.
 \end{aligned}
 \tag{3.26}$$

From this representation and the arbitrariness of  $\xi_0 \in W$ , (3.6) follows immediately by applying Theorem (7) and Corollary (1).

This concludes the sketch of the proof of Theorem (1).  $\square$

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